

Fermionic impurities in the unquenched ABJM

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ABSTRACT: We study, in a holographic setup, the effect of adding localized fermionic impurities to the three-dimensional Chern–Simons-matter theories with unquenched fields in the fundamental representation of the gauge group. The impurities are introduced as probe D6-branes extending along the radial direction and wrapping a five-dimensional submanifold inside a squashed \mathbb{CP}^3 . We analyze the straight flux tube embeddings and study the corresponding fluctuation modes of the D6-branes. The conformal dimensions of the operators dual to such fluctuations depend non-trivially on the ratio of the flavor number to the Chern–Simon level of the unquenched ABJM.

KEYWORDS: Gauge-gravity correspondence

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1 Introduction

One of the developments emerging from string theory explorations is the idea of a gauge/gravity correspondence [1]. The remarkable feature of the correspondence is the relation of the strongly coupled regime of the gauge theory to the weakly coupled regime of the string theory and vice-versa. Consequently, it has become a powerful tool in studying strongly interacting systems, allowing for novel computations that go beyond the standard perturbative techniques of quantum field theories.

In this theme, the recent developments on the AdS_4/CFT_3 constitute a rich framework in which fundamental questions about the correspondence can be posted. In particular, the ABJM theory [2] is a $U(N) \times U(N)$ Chern–Simons gauge theory with levels $(k, -k)$ and bifundamental matter fields. In the large N limit the theory acquires a supergravity description in terms of the $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$ geometry. When the Chern–Simons level is large the size of the fiber is small and the system acquires a ten-dimensional description in terms of $AdS_4 \times \mathbb{CP}^3$ with fluxes, that preserves 24 supersymmetries. The addition of flavor to the ABJM theory (fields transforming in the fundamental representations $(N, 1)$ and $(1, N)$ of the $U(N) \times U(N)$ gauge group) is realized through D6-branes filling the AdS_4 space and wrapping a submanifold inside the \mathbb{CP}^3 , while preserving a fraction of the initial supersymmetry [3]. The addition of a large number of flavor branes, continuously smeared in the transverse space, produces a backreaction of the original geometry and induces a deformation. Utilizing techniques developed in [4, 5] and reviewed in [6], this unquenched solution was computed in [7] and depends non-trivially on the number of flavors.

The addition of an extra set of branes interacting with the colored ones creates a defect in the gauge theory. The characteristic example of this class is $\mathcal{N} = 4$ Super-Yang-Mills (SYM) with fermionic impurities. The branes realization of this system is obtained by adding D5-branes to the $AdS_5 \times S^5$ background, dual to $\mathcal{N} = 4$ SYM. The D5-branes are extended in the holographic direction and wrap an \mathbb{S}^4 inside the \mathbb{S}^5 in such a way that their worldvolume is an $AdS_2 \times \mathbb{S}^4$ submanifold of $AdS_5 \times \mathbb{S}^5$ [8] (for other embeddings with lower dimensional spheres see [9]). More recently, these configurations have been used in [10, 11] to construct holographic dimer models, by considering D-branes which connect two spatially separated impurities on the boundary of AdS . The holographic setup of D3- and D5-branes also realizes the maximally supersymmetric Kondo model [12–14] (see also [15, 16]).

In this paper we consider D6-branes extending along an $AdS_2 \subset AdS_4$ and a wrapping of a five-dimensional submanifold inside the squashed \mathbb{CP}^3 , in order to construct the holographic dual of a Chern–Simons-matter theory with flavor and fermionic impurities. We will build our solution on the unquenched background solution [7].

An overview of the paper is as follows: In section 2 we give a short, self contained, review of the gravity dual of a three-dimensional Chern–Simons-matter theory with unquenched fields in the fundamental representation of the gauge group. In section 3 we introduce impurities as probe D6-branes wrapping a five-dimensional submanifold inside the squashed \mathbb{CP}^3 and analyze straight flux tube embeddings. In section 4 we analyze in detail the fluctuation modes of the D6-branes around the straight flux tube configurations and compute the conformal dimensions of the operators dual to such fluctuations. Due to the presence on the unquenched ABJM there is an explicit dependence on the number of flavors. In section 5 we conclude and discuss lines of possible future related research. In the appendix A we analytically derive the Lagrangian for the fluctuations of the probe brane and in appendix B the explicit solution of the radial type differential eq. arising in these fluctuations. In appendix C we compute the spectrum of the Laplacian corresponding to the angular part of the operator entering into the fluctuation analysis.

2 Review of the ABJM with unquenched massless flavor

In this section, following [7], we will review the solutions of type-IIA supergravity which are dual to three-dimensional Chern–Simons-matter theories with unquenched fields in the fundamental representation of the gauge group. The addition of flavor is performed by means of D6-branes that are extended along the Minkowski gauge theory directions and are delocalized in the internal space in such a way that the system is $\mathcal{N} = 1$ supersymmetric and the flavor group is Abelian. The geometry has the form of a product space $AdS_4 \times \mathcal{M}_6$, where \mathcal{M}_6 is a six-dimensional compact manifold whose metric is obtained by squashing the Fubini–Study metric of \mathbb{CP}^3 [17] with suitable constant factors depending on the number of flavors.

The metric of the flavored ABJM background (in the string frame) is given by

$$ds^2 = L^2 ds_{AdS_4}^2 + ds_6^2, \quad (2.1)$$

with the standard parametrization for the AdS_4 metric

$$ds_{AdS_4}^2 = r^2 (-dt^2 + dx^2 + dy^2) + \frac{dr^2}{r^2}, \quad (2.2)$$

while the six-dimensional metric is written in terms of the $SU(2)$ instanton of S^4

$$ds_6^2 = \frac{L^2}{b^2} \left[q ds_{\mathbb{S}^4}^2 + (dx^i + \epsilon^{ijk} A_j x^k)^2 \right], \quad (2.3)$$

where b and q are constant squashing factors. The metric for the unit S^4 is denoted by $ds_{\mathbb{S}^4}^2$ and x^i ($i = 1, 2, 3$) are Cartesian coordinates parametrizing the unit S^2 whereas A^i , $i = 1, 2, 3$ are the components of the non-Abelian one-form connection corresponding to the $SU(2)$ instanton. The solution depends on two integers N and k which, on the gauge theory side, represent the rank of the gauge group and the Chern–Simons level, respectively. In string units, the AdS_4 radius L can be written as

$$L^4 = 2\pi^2 \frac{N}{k} \frac{(2-q)b^4}{q(q+\eta q - \eta)}. \quad (2.4)$$

Introducing a set of $SU(2)$ left-invariant one-forms, which satisfy the usual relation $d\omega_i = \frac{1}{2} \epsilon_{ijk} \omega_j \wedge \omega_k$, together with the coordinate α ($0 \leq \alpha \leq \pi$) one parametrizes the unit S^4 as

$$ds_{\mathbb{S}^4}^2 = d\alpha^2 + \frac{\sin^2 \alpha}{4} (\omega_1^2 + \omega_2^2 + \omega_3^2). \quad (2.5)$$

The parameterization for the ω 's is

$$\begin{aligned} \omega_1 &= \cos \psi d\theta_2 + \sin \psi \sin \theta_2 d\phi_2 \\ \omega_2 &= \sin \psi d\theta_2 - \cos \psi \sin \theta_2 d\phi_2 \\ \omega_3 &= d\psi + \cos \theta_2 d\phi_2, \end{aligned} \quad (2.6)$$

while the $SU(2)$ instanton one-forms A_i are given by

$$A_i = -\sin^2 \frac{\alpha}{2} \omega_i. \quad (2.7)$$

Parametrizing the x^i coordinates of the \mathbb{S}^2 by means of the angles θ_1 and ϕ_1 , the following relation holds¹

$$(dx^i + \epsilon^{ijk} A_j A_k)^2 = E_1^2 + E_2^2, \quad (2.8)$$

where E_1 and E_2 are the following one-forms

$$\begin{aligned} E_1 &= d\theta_1 + \sin^2 \frac{\alpha}{2} (\omega_1 \sin \phi_1 - \omega_2 \cos \phi_1) \\ E_2 &= \sin \theta_1 \left(d\phi_1 - \omega_3 \sin^2 \frac{\alpha}{2} \right) + \sin^2 \frac{\alpha}{2} \cos \theta_1 (\omega_1 \cos \phi_1 + \omega_2 \sin \phi_1). \end{aligned} \quad (2.9)$$

¹Explicitly we have $x^1 = \sin \theta_1 \cos \phi_1$, $x^2 = \sin \theta_1 \sin \phi_1$, $x^3 = \cos \theta_1$ with $0 \leq \theta_1 < \pi$ and $0 \leq \phi_1 < 2\pi$. For completeness we also note that $0 \leq \theta_2 < \pi$, $0 \leq \phi_2 < 2\pi$ and $0 \leq \psi < 4\pi$.

Putting all these ingredients together we rewrite the six-dimensional metric (2.3) as

$$ds_6^2 = \frac{L^2}{b^2} \left[q ds_{\mathbb{S}^4}^2 + E_1^2 + E_2^2 \right]. \quad (2.10)$$

In order to write the expression for the F_2 we will introduce a new set of one-forms

$$\begin{aligned} S_1 &= \sin \phi_1 \omega_1 - \cos \phi_1 \omega_2, \\ S_2 &= \sin \theta_1 \omega_3 - \cos \theta_1 (\cos \phi_1 \omega_1 + \sin \phi_1 \omega_2), \\ S_3 &= -\cos \theta_1 \omega_3 - \sin \theta_1 (\cos \phi_1 \omega_1 + \sin \phi_1 \omega_2). \end{aligned} \quad (2.11)$$

Then, the ansatz for the F_2 is the following

$$F_2 = \frac{k}{2} \left[E_1 \wedge E_2 - \eta (\mathcal{S}_\alpha \wedge \mathcal{S}_3 + \mathcal{S}_1 \wedge \mathcal{S}_2) \right], \quad (2.12)$$

where the one-forms \mathcal{S}_α and \mathcal{S}_i are

$$\mathcal{S}_\alpha = d\alpha, \quad \mathcal{S}_i = \frac{\sin \alpha}{2} S_i, \quad i = 1, 2, 3 \quad (2.13)$$

and η is a squashing parameter, directly related to the number N_f of flavors as

$$\eta = 1 + \frac{3N_f}{4k}, \quad 1 \leq \eta < \infty. \quad (2.14)$$

The internal squashing q is related to η through the relation

$$q = \frac{3(1 + \eta) - \sqrt{9\eta^2 - 2\eta + 9}}{2}, \quad (2.15)$$

while b can be written in terms of q and η as

$$b = \frac{q(\eta + q)}{2(q + \eta q - \eta)}. \quad (2.16)$$

The dilaton and the F_4 have the following expressions

$$e^{-\Phi} = \frac{b}{4} \frac{\eta + q}{2 - q} \frac{k}{L}, \quad F_4 = \frac{3k}{4} \frac{(\eta + q)b}{2 - q} L^2 \Omega_{AdS_4}. \quad (2.17)$$

The range of $q \in [1, 5/3]$, the value $q = 1$ (no flavors) corresponds to the $\mathcal{N} = 6$ ABJM background. We also note that had we taken the positive square root in (2.15) it would have corresponded to a different branch for which $q \geq 5$. For the value $q = 5$ the background has reduced supersymmetry, whose metric is the sum of the AdS_4 metric and the squashed \mathbb{CP}^3 . The latter metric is not the one corresponding to an Einstein space [18, 19]. This would have corresponded to the value $q = 2$, which nevertheless is not allowed in either branch.

3 The Hamiltonian density of D6-brane probes

In this section we will consider a D6-brane probe extending along the radial direction r and wrapping a five-dimensional submanifold inside the squashed \mathbb{CP}^3 at constant values of the spatial Minkowski directions x and y . The background coordinates X^M and the worldvolume coordinates ζ^μ are

$$\begin{aligned} \text{Background: } X^M &= (t, x, y, r, \alpha, \theta_2, \phi_2, \psi, \theta_1, \phi_1) , \\ \text{Brane: } \zeta^\mu &= (t, r, \gamma^i) = (t, r, \theta_2, \phi_2, \psi, \theta_1, \phi_1) . \end{aligned} \tag{3.1}$$

We consider embeddings in which the angle α depends only on the radial direction, i.e.

$$\alpha = \alpha(r) . \tag{3.2}$$

These embeddings correspond to configurations in which the flux tube starts from the boundary of the AdS_4 and reaches the origin of the holographic coordinate.² We will also turn on an electric worldvolume gauge field component F_{0r} , whose source is the RR potential C_5 through the Wess–Zumino (WZ) term of the D6-brane action.

The Dirac-Born-Infeld (DBI) part of the D6-brane action is given by

$$\mathcal{S}_{DBI} = -T_6 \int d^7\zeta e^{-\Phi} \sqrt{-\det(g + F)} , \tag{3.3}$$

where g is the induced metric on the worldvolume of the D6-brane and T_6 is the brane tension. After integrating over all the angles of the internal space we arrive at the following expression for the DBI contribution to the action

$$S_{DBI} = \int dt dr \mathcal{L}_{DBI} , \tag{3.4}$$

with

$$\mathcal{L}_{DBI} = -\frac{NL^2}{8\pi} \frac{b}{\sqrt{q}} \sin^3 \alpha \sqrt{1 + \frac{q}{b^2} r^2 \alpha'^2 - L^{-4} F_{0r}^2} , \tag{3.5}$$

where α' denotes $d\alpha/dr$. The WZ part of the action is given by

$$\mathcal{L}_{WZ} = T_6 \int C_5 \wedge F \equiv \int dt dr \mathcal{L}_{WZ} . \tag{3.6}$$

²That excludes hanging flux tubes, namely configurations that reach a minimal value of r and return to the boundary. In such cases a non-constant Cartesian coordinate for the embedding is needed.

In type-IIA supergravity the RR six-form is defined as the Hodge dual of the RR four-form, namely $\star F_4 = -F_6$. Through that relation we obtain the expression for the five-form potential C_5 as

$$C_5 = -\frac{\pi^2}{8} N C(\alpha) \sin \theta_1 \sin \theta_2 d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \wedge d\psi, \quad (3.7)$$

where

$$C(\alpha) = \cos \alpha (\sin^2 \alpha + 2) - 2. \quad (3.8)$$

After integrating over the above five angles γ^i we obtain

$$\mathcal{L}_{WZ} = -\frac{N}{8\pi} C(\alpha) F_{0r}. \quad (3.9)$$

The total Lagrangian density is given by the sum of the (3.5) and (3.9)

$$\mathcal{L} = -\frac{N}{8\pi} \frac{b}{\sqrt{q}} \left[L^2 \sin^3 \alpha \sqrt{1 + \frac{q}{b^2} r^2 \alpha'^2} - L^{-4} F_{0r}^2 + \frac{\sqrt{q}}{b} C(\alpha) F_{0r} \right]. \quad (3.10)$$

The equation of motion for the gauge field derived from (3.10) implies that

$$\frac{\partial \mathcal{L}}{\partial F_{0r}} = \text{constant}. \quad (3.11)$$

Using the quantization condition derived in [8] one relates this constant to the number n of strings (quarks) of the flux tube as

$$\frac{\partial \mathcal{L}}{\partial F_{0r}} = n T_f, \quad (3.12)$$

where T_f is the tension of the fundamental string and $n \in \mathbb{Z}$ is the fundamental string charge carried by the D6-brane. Exploiting the above quantization condition, we obtain

$$\frac{\sin^3 \alpha}{\sqrt{1 - L^{-4} F_{0r}^2 + \frac{q}{b^2} r^2 \alpha'^2}} = \frac{\sqrt{\sin^6 \alpha + C_n(\alpha)^2}}{\sqrt{1 + \frac{q}{b^2} r^2 \alpha'^2}}, \quad (3.13)$$

where we have defined

$$C_n(\alpha) \equiv \frac{\sqrt{q}}{b} \left(C(\alpha) + \frac{4n}{N} \right). \quad (3.14)$$

Then from (3.13) we obtain for the field strength

$$F_{0r} = \frac{L^2 \sqrt{1 + \frac{q}{b^2} r^2 \alpha'^2}}{\sqrt{\sin^6 \alpha + C_n(\alpha)^2}} C_n(\alpha). \quad (3.15)$$

In order to eliminate the electric field from the equations of motion we compute the Hamiltonian of the system by performing a Legendre transformation in (3.10)

$$\mathcal{H} = F_{0r} \frac{\partial \mathcal{L}}{\partial F_{0r}} - \mathcal{L}. \quad (3.16)$$

Using the above results together with (3.13) we end up with the following formula for the Hamiltonian density

$$\mathcal{H} = \frac{NL^2}{8\pi} \frac{b}{\sqrt{q}} \sqrt{1 + \frac{q}{b^2} r^2 \alpha'^2} \sqrt{\sin^6 \alpha + C_n(\alpha)^2}. \quad (3.17)$$

It remains to determine $\alpha(r)$ by integrating the corresponding Euler-Lagrange equations. In the next subsection we will constrain our analysis to configurations with constant α . Embeddings with α depending on the holographic coordinate are related to the baryon vertex of the ABJM theory and will be analyzed in future work (for the similar analysis in the $AdS_5 \times S^5$ case see [20, 21]).

3.1 Flux tube configurations

In this subsection we will calculate the energy density of the configurations with constant α . Such configurations must satisfy the condition

$$\left. \frac{\partial \mathcal{H}}{\partial \alpha} \right|_{\alpha'=0} = 0. \quad (3.18)$$

Since

$$\left. \frac{\partial \mathcal{H}}{\partial \alpha} \right|_{\alpha'=0} = 3 \frac{NL^2}{8\pi} \frac{b}{\sqrt{q}} \frac{\sin^3 \alpha \Lambda_n(\alpha)}{\sqrt{\sin^6 \alpha + C_n(\alpha)^2}}, \quad (3.19)$$

with

$$\Lambda_n(\alpha) \equiv \sin^2 \alpha \cos \alpha - \frac{\sqrt{q}}{b} C_n(\alpha), \quad (3.20)$$

the non-trivial configurations with constant α are the solutions of the following algebraic equation

$$\Lambda_n(\alpha) = 0, \quad (3.21)$$

which can be written as a cubic equation in $\cos \alpha$ as

$$\left(1 - \frac{q}{b^2}\right) \cos^3 \alpha - \left(1 - \frac{3q}{b^2}\right) \cos \alpha - \frac{2q}{b^2} \left(1 - \frac{2n}{N}\right) = 0. \quad (3.22)$$

Due to Bolzano's theorem³ the function $\Lambda_n(\alpha)$ has at least one root in the interval $\alpha \in [0, \pi]$ for every n in the range $0 \leq n \leq N$, while the monotonicity of the function in this interval tells us that the root is unique. After using (3.20) to express $C_n(\alpha_n)$ in terms of α_n

$$C_n(\alpha_n) = \frac{b}{\sqrt{q}} \sin^2 \alpha_n \cos \alpha_n, \quad (3.24)$$

as well as (3.17), we obtain the energy density of the configurations with constant α

$$E_n = \frac{NL^2}{8\pi} \frac{b^2}{q} \sin^2 \alpha_n \sqrt{\cos^2 \alpha_n + \frac{q}{b^2} \sin^2 \alpha_n}, \quad (3.25)$$

where α_n is a solution of (3.22).⁴ The constant electric field F_{0r} corresponding to such configurations is computed from (3.15). One finds that

$$\bar{f}_{0r} = \frac{L^2 \cos \alpha_n}{\sqrt{\cos^2 \alpha_n + \frac{q}{b^2} \sin^2 \alpha_n}}. \quad (3.26)$$

Notice that from (3.22) we have

$$\Lambda_n(\alpha_n) = -\Lambda_{N-n}(\pi - \alpha_n) \quad \implies \quad \alpha_{N-n} = \pi - \alpha_n, \quad (3.27)$$

which combined with (3.25), is telling us that E_n is invariant under the change $n \rightarrow N-n$, as it should if an object is to transform in the fully anti-symmetric representation of the gauge group, with n being the number of boxes in the corresponding Young tableaux. The induced metric on the D6-brane worldvolume is

$$ds^2 = L^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + ds_{M_5}^2 \right], \quad (3.28)$$

which is of the form $AdS_2 \times M_5$, with the line element of M_5 having the following expression

$$ds_{M_5}^2 = \tilde{g}_{ij} d\gamma^i d\gamma^j = \frac{q}{4b^2} \sin^2 \alpha_n (\omega_1^2 + \omega_2^2 + \omega_3^2) + \frac{1}{b^2} (E_1^2 + E_2^2). \quad (3.29)$$

Under the change of α_n described in (3.27) the line element of (3.29) remains invariant as well.

³Note that $q/b^2 \in [1, \frac{16}{15}]$, monotonously as $\eta \in [1, \infty)$ and that

$$\Lambda_n(0) = -4 \frac{q}{b^2} \frac{n}{N} \leq 0, \quad \Lambda_n(\pi) = 4 \frac{q}{b^2} \left(1 - \frac{n}{N}\right) \geq 0. \quad (3.23)$$

⁴For $n \ll N$ the energy turns out to be simply the sum of the energies of the individual fundamental strings, i.e. $E_n \simeq \frac{n}{2\pi} L^2$, where the extra factor L^2 is related to the overall appearance of the same factor in (2.1) due to a rescaling of the world-volume coordinates. It is worth noting that in this dilute target approximation there is no dependence of the energy on the flavor number.

4 Fluctuations of the impurities

Moving one step forward, we will study in this section fluctuations around the static configurations we have computed. For this reason we consider the following

$$\alpha = \alpha_n + \xi(\zeta), \quad F = \bar{f} + f(\zeta), \quad x = \bar{x} + \chi(\zeta), \quad (4.1)$$

where α_n is a solution of the condition $\Lambda_n(\alpha) = 0$, \bar{x} is the constant Cartesian coordinate of the unperturbed D6-brane and \bar{f} is the background gauge field strength with non-vanishing component given by (3.26). The fluctuations around these constant values, namely ξ , χ and f , depend, as indicated, on the D-brane coordinates ζ^μ in (3.1). The total perturbed Lagrangian density is the sum of the DBI and the WZ parts, and a detailed derivation is presented in the appendix A. Indeed, if in (A.23) we neglect constant and total derivative terms we find the following Lagrangian density for the quadratic fluctuations

$$\begin{aligned} \mathcal{L} = & -T_6 \frac{\pi^2 N b^6 L^2}{q^2} P^{1/2} \sqrt{\tilde{g}} \left\{ \frac{1}{2} L^2 r^2 \mathcal{G}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{1}{2} \frac{q}{b^2} L^2 \mathcal{G}^{\mu\nu} \partial_\mu \xi \partial_\nu \xi \right. \\ & \left. + \frac{1}{4} \mathcal{G}^{\mu\rho} \mathcal{G}^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma} + V \xi^2 - W \xi f_{0r} \right\}, \end{aligned} \quad (4.2)$$

where we have for notational convenience set

$$\begin{aligned} P &= \frac{\sin^6 \alpha_n}{\sin^6 \alpha_n + C_n(\alpha_n)^2}, \quad V = -\frac{3}{2 \sin^2 \alpha_n}, \\ W &= \frac{3}{L^2 P^{1/2} \sin^3 \alpha_n} \left(C_n(\alpha_n) \cot \alpha_n + \frac{\sqrt{q}}{b} \sin^3 \alpha_n \right) \end{aligned} \quad (4.3)$$

and the seven-dimensional metric \mathcal{G} is defined in (A.7).

4.1 Fluctuation of the Cartesian coordinate

Here we study the fluctuations of the Cartesian coordinate which do not couple to those of the gauge field and of the embedding angular coordinate. The equation of motion for χ can be easily derived from (4.2) and it is

$$\partial_r(r^4 \partial_r \chi) - \partial_0^2 \chi + r^2 P \nabla_{M_5}^2 \chi = 0. \quad (4.4)$$

To obtain the above equation of motion we explicitly used the components of $\mathcal{G}^{\mu\nu}$, while $\nabla_{M_5}^2$ is the Laplacian operator of the five-dimensional manifold in (3.29) (see appendix

C). The actual expression for this operator is quite complicated. Remarkably, we were able to find explicit solutions of the form either $\chi = \chi(t, r, \theta_2, \phi_2, \psi)$ or $\chi = \chi(t, r, \theta_1, \phi_1)$ (see appendices C.1 and C.2, respectively). Without loss of generality in the following analysis of the conformal dimensions we will focus on the first class of solutions. Using the separation of variables

$$\chi = e^{iEt} R(r) \Omega(\theta_2, \phi_2, \psi), \quad (4.5)$$

and equation (C.5) from appendix C.1

$$\nabla_{M_5}^2 \Omega = \frac{b^2}{q \sin^2 \alpha_n} \nabla_{S^3} \Omega = -\frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} \Omega, \quad l = 0, 1, 2, \dots \quad (4.6)$$

the equation of motion for the radial function $R(r)$ becomes

$$\partial_r (r^4 \partial_r R) + \left(E^2 - \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} P r^2 \right) R = 0. \quad (4.7)$$

This equation can be solved exactly, but since we are interested in the asymptotic behavior of the solution we put all the details on the analytic derivation in appendix B. Assuming that $R(r) \sim r^\lambda$ at large r , we arrive to the following quadratic equation

$$\lambda(\lambda + 3) = \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} P, \quad (4.8)$$

with solutions that we will denote as λ_1 and λ_2 . We would like to associate them with the dimensions Δ of operators on the defect theory. Since the fluctuations are not canonically normalized since there is a factor r^2 in front of the kinetic term for the field χ in (4.2). Hence, we cannot simply use the usual relation [23]

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}, \quad (4.9)$$

with $d = 1$. Instead one may employ an approach that gives the result immediately [24] According to this, if a scalar field in AdS_2 at large r behaves as

$$\chi \sim d_1 r^{-2\lambda_1} + d_2 r^{-2\lambda_2}, \quad \lambda_2 > \lambda_1, \quad (4.10)$$

then the dimension of the operator dual to the normalizable mode is

$$\Delta = \frac{1}{2} + \lambda_2 - \lambda_1. \quad (4.11)$$

In our case the conformal dimension becomes

$$\Delta = \frac{1}{2} + \sqrt{\frac{9}{4} + \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n}} P = \frac{1}{2} + \sqrt{\frac{9}{4} + \frac{l(l+2)}{q/b^2 \sin^2 \alpha_n + \cos^2 \alpha_n}}. \quad (4.12)$$

In general, the conformal dimension is not a rational number and $\Delta = 2$ for $l = 0$. Moreover the dimension depends on the filling fraction ν , through the dependence of the angle α_n , and is invariant under the change of (3.27). Also the range of the parameter q/b^2 in footnote 2, the prefactor multiplying $l(l+2)$ is of order one.⁵ The dependence of the conformal dimension on the number of flavors becomes is in general complicated and it admits a power series expansion around the unquenched result. In particular, in the half filling fraction case, $\nu = \frac{1}{2}$ the cubic equation (3.22) has the following solution in the interval $\alpha_n \in [0, \pi]$

$$\cos \alpha_n = 0 \quad \Rightarrow \quad \alpha_n = \frac{\pi}{2}. \quad (4.13)$$

Expanding (4.12) around the unquenched result we have

$$\Delta = \frac{1}{2} + \sqrt{\frac{9}{4} + l(l+2)} - \frac{9}{512} \frac{l(l+2)}{\sqrt{\frac{9}{4} + l(l+2)}} \left(\frac{N_f}{k} \right)^2 + \mathcal{O} \left(\frac{N_f}{k} \right)^3. \quad (4.14)$$

4.2 Coupled modes

In this subsection we will focus our attention on the fluctuations of the gauge and scalar fields, which through their equations of motion appear to be coupled. The equation of motion for the gauge field is given by

$$\frac{1}{\sqrt{\bar{g}}} \partial_\rho \left(\sqrt{\bar{g}} \mathcal{G}^{\mu\rho} \mathcal{G}^{\nu\sigma} f_{\mu\nu} \right) + W \left(\partial_r \xi \delta_0^\sigma - \partial_0 \xi \delta_r^\sigma \right) = 0, \quad (4.15)$$

while that for the scalar is

$$\frac{1}{\sqrt{\bar{g}}} \partial_\mu \left(\sqrt{\bar{g}} \mathcal{G}^{\mu\nu} \partial_\nu \xi \right) - \frac{2}{L^2} \frac{b^2}{q} V \xi + \frac{b^2}{q} \frac{1}{L^2} W f_{0r} = 0. \quad (4.16)$$

We consider the following ansatz for the fluctuations of the gauge fields and the scalar (the only non-vanishing components of the gauge field are \hat{A}_r and \hat{A}_i)

$$\begin{aligned} \hat{A}_r &= e^{iEt} \Omega(\theta_2, \phi_2, \psi) \phi(r), & \hat{A}_i &= e^{iEt} \partial_i \Omega(\theta_2, \phi_2, \psi) \tilde{\phi}(r) \\ \xi_r &= e^{iEt} \Omega(\theta_2, \phi_2, \psi) z(r). \end{aligned} \quad (4.17)$$

⁵Another way to arrive at the same result is to redefine the fluctuations by absorbing the factor r in the kinetic for χ in (4.2) into a new scalar field $\phi = r\chi$. Then ϕ becomes canonically normalized but after some algebraic manipulations one sees that m^2 is shifted by a factor of 2. Then after using (4.9), with $d = 1$, one derives (4.12).

Note that the fact that the vector index of \hat{A}_i is due to the derivative on Ω is, as it turns out, for consistency of the ansatz upon substitution into the above equations of motion. Then, from (4.15) for $\sigma = 0$ we obtain

$$\frac{b^2 l(l+2)}{q \sin^2 \alpha_n} \tilde{\phi} = \frac{r^2}{P} \phi' - \frac{\imath}{E} L^4 P W r^2 z', \quad (4.18)$$

while if we set $\sigma = r$ in the same equation we have that

$$\frac{E^2}{L^4 P^2} \phi + \frac{b^2 l(l+2)}{q \sin^2 \alpha_n} \frac{r^2}{L^4 P} (\tilde{\phi}' - \phi) - \imath E W z = 0. \quad (4.19)$$

Also from (4.15) for $\sigma = i$ we have

$$E^2 \tilde{\phi} = -r^2 \frac{d}{dr} \left[r^2 (\tilde{\phi}' - \phi) \right], \quad (4.20)$$

an equation that can be easily derived from (4.18) and (4.19). Using (4.18) in order to eliminate $\tilde{\phi}$ from (4.19), we have

$$\frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) + \frac{E^2}{r^2} \phi - \frac{b^2 l(l+2)}{q \sin^2 \alpha_n} P \phi - \imath \frac{L^4}{E} P^2 W \left\{ \frac{d}{dr} \left(r^2 \frac{dz}{dr} \right) + \frac{E^2}{r^2} z \right\} = 0, \quad (4.21)$$

while the equation of motion for the scalar, using (4.17), becomes

$$\frac{d}{dr} \left(r^2 \frac{dz}{dr} \right) + \frac{E^2}{r^2} z - \frac{b^2}{q} \left(2V + \frac{l(l+2)}{\sin^2 \alpha_n} \right) P z + \imath \frac{b^2}{q} E P W \phi = 0. \quad (4.22)$$

At this point, we define the differential operator $\hat{\mathcal{O}}$, which acts on functions as follows

$$\hat{\mathcal{O}} f = \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) + \frac{E^2}{r^2} f \quad (4.23)$$

and make the following field redefinitions

$$\hat{z} = \imath \frac{L^4}{E} P^2 W z, \quad \eta = \phi - \imath \frac{L^4}{E} P^2 W z = \phi - \hat{z}. \quad (4.24)$$

Then (4.22) and (4.25) can be written in a more compact form as

$$\left(\hat{\mathcal{O}}_{\mathbb{I}_{2 \times 2}} - \mathcal{M} \right) \begin{pmatrix} \hat{z} \\ \eta \end{pmatrix} = 0, \quad (4.25)$$

where the entries of the matrix \mathcal{M} are

$$\begin{aligned} \mathcal{M}_{11} &= \frac{b^2}{q} \left(2V + L^4 W^2 P^2 + \frac{l(l+2)}{\sin^2 \alpha_n} \right) P \\ \mathcal{M}_{12} &= \frac{b^2}{q} L^4 W^2 P^3, \quad \mathcal{M}_{21} = \mathcal{M}_{22} = \frac{b^2 l(l+2)}{q \sin^2 \alpha_n} P, \end{aligned} \quad (4.26)$$

while its eigenvalues are

$$\lambda_{\pm} = \frac{P}{2} \frac{b^2}{q} \left[2V + L^4 P^2 W^2 + 2 \frac{l(l+2)}{\sin^2 \alpha_n} \pm \sqrt{4 \frac{l(l+2)}{\sin^2 \alpha_n} L^4 P^2 W^2 + (2V + L^4 P^2 W^2)^2} \right]. \quad (4.27)$$

Now, if ψ_{\pm} are the eigenvectors of \mathcal{M} with eigenvalues λ_{\pm} , then (4.25) takes the form

$$\frac{d}{dr} \left(r^2 \frac{d\psi_{\pm}}{dr} \right) + \left(\frac{E^2}{r^2} - \lambda_{\pm} \right) \psi_{\pm} = 0, \quad (4.28)$$

and in order to study the behavior of ψ_{\pm} at large r we assume that $\psi_{\pm} \sim r^s$. Then from the above differential equation we find that s should obey the following quadratic equation

$$s(s+1) - \lambda_{\pm} = 0, \quad (4.29)$$

with solutions

$$s_{\pm} = \frac{-1 \pm \sqrt{1 + 4\lambda_{\pm}}}{2}, \quad s_- < s_+. \quad (4.30)$$

Noting that in the kinetic term in (4.2) for these type of fluctuations there is no overall extra overall factor of r , we may safely use for the conformal dimension the expression [23] with $d = 1$ and $m^2 = \lambda_+$, i.e.

$$\Delta = \frac{1}{2} \left(1 + \sqrt{1 + 4\lambda_+} \right). \quad (4.31)$$

In general, it is not a rational number and depends on the filling fraction ν . Notice, that unlike (4.12), for $l = 0$ the conformal dimension Δ_+ does depend on the flavor number. As also argued in the appendix C, for angular dependence of the form $\Omega(\theta_1, \phi_1)$ we may use the previous results by simply performing the replacement (C.8). In the half filling fraction case $\nu = \frac{1}{2}$ and expanding (4.31) around the unquenched result we obtain that

$$\Delta = 3 + l - \frac{9}{512} \frac{(l+3)(l+2)(2l-1)}{(l+1)(5+2l)} \left(\frac{N_f}{k} \right)^2 + \mathcal{O} \left(\frac{N_f}{k} \right)^3 \quad (4.32)$$

5 Conclusions and future directions

In this paper we studied localized fermionic impurities to the unquenched ABJM Chern–Simons–matter theory, which are realized through the addition of fields transforming in the fundamental representations $(N, 1)$ and $(1, N)$ of the $U(N) \times U(N)$ gauge

group. In the holographic approach the impurities are added by introducing probe D6-branes, extending along the holographic coordinate and wrapping a five-dimensional submanifold inside a squashed \mathbb{CP}^3 at constant values of the Minkowski directions. The background RR field induces an electric gauge-field on the world-volume of the probe branes, giving rise to a bundle of strings that form a flux tube which prevents the collapse of the wrapped brane.

We concentrated on the simplest of the configurations in which the flux tube starts from the boundary of the AdS_4 and reach the origin of the holographic coordinate. More general solutions including the baryon vertex of the unquenched ABJM theory and hanging flux tubes, namely configurations that reach a minimal value in the holographic coordinate and return to the boundary, are left as open problems for future work.

The natural step forward was to investigate of the stability for the probe D6-branes, that introduce the holographic impurities. We presented an analytic study for the fluctuations of those probes in the unquenched ABJM. The fluctuations are separated in two categories. The first contains just the decoupled fluctuations of the Cartesian coordinates while the second the coupled fluctuations of the angular embedding function and the world-volume gauge field. The coupled modes were shown to decouple by appropriate field redefinitions. In this way we were able to determine the spectrum of conformal dimensions of the dual operators in the defect theory. The novel feature of our analysis is that using the unquenched ABJM background we obtained expressions of the conformal dimension that explicitly depend on the number of fundamental flavors, thus generalizing the previously obtained results for the ABJM background [11].

There are many interesting questions that follow from the analysis we presented in the recent paper, and we would like to pursue some of them in the near future. In the quenched ABJM background there is robust proposition in [22] that the Wilson lines in the antisymmetric representations of the gauge group of the ABJM are holographically realized by D6-branes extending along an $AdS_2 \subset AdS_4$ and wrapping a five-dimensional submanifold of \mathbb{CP}^3 . These are the natural brane configurations to consider in order to construct the holographic dual of the Chern–Simons-matter theory with fermionic impurities. Unlike to the quenched case, there is no such proof for the unquenched ABJM, though we note the invariance under $n \rightarrow N - n$ in (3.27). It would be very interesting to pursue this issue further.

The addition of fermionic impurities in the unquenched ABJM background at finite temperature [25] will create a much richer structure. The analysis of the thermodynamic properties, of both straight and hanging flux tubes, is expected to unveil a competition between the two configurations. This in turn will lead to the existence of a dimerization transition similar to the one presented in [10, 11], but now in a background that will include the non-trivial presence of fundamental flavors.

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A Fluctuation analysis

In this appendix we will analyze the small perturbations around the flux tube configurations, derived in section 3.1. We will analytically obtain the second order lagrangian for those fluctuations, which is the starting point of section 4.

We perturb a D6-brane probe as in (4.1) and expand the DBI+WZ action to second order in the perturbations ξ , f and χ . Starting with the components of the perturbed induced metric we write

$$g = \bar{g} + \hat{g}, \quad (\text{A.1})$$

where \bar{g} is the zeroth order induced metric and the perturbation \hat{g} has the form

$$\hat{g}_{\mu\nu} = L^2 \left[r^2 \partial_\mu \chi \partial_\nu \chi + \frac{q}{b^2} \partial_\mu \xi \partial_\nu \xi + \xi \hat{g}_{\mu\nu}^{(1)} + \xi^2 \hat{g}_{\mu\nu}^{(2)} \right] + \dots, \quad (\text{A.2})$$

where the $\hat{g}_{\mu\nu}^{(1)}$ and $\hat{g}_{\mu\nu}^{(2)}$ are given by (their indices take values only in the angular part)

$$\begin{aligned}\hat{g}_{ij}^{(1)} d\gamma^i d\gamma^j &= \frac{q}{4b^2} \sin 2\alpha_n (\omega_1^2 + \omega_2^2 + \omega_3^2) + \frac{2}{b^2} \left[E_1 \frac{\partial E_1}{\partial \alpha} + E_2 \frac{\partial E_2}{\partial \alpha} \right] \Big|_{\alpha=\alpha_n}, \\ \hat{g}_{ij}^{(2)} d\gamma^i d\gamma^j &= \frac{q}{4b^2} \cos 2\alpha_n (\omega_1^2 + \omega_2^2 + \omega_3^2) \\ &\quad + \frac{1}{b^2} \left[\left(\frac{\partial E_1}{\partial \alpha} \right)^2 + \left(\frac{\partial E_2}{\partial \alpha} \right)^2 + E_1 \frac{\partial^2 E_1}{\partial \alpha^2} + E_2 \frac{\partial^2 E_2}{\partial \alpha^2} \right] \Big|_{\alpha=\alpha_n}.\end{aligned}\tag{A.3}$$

The determinant of the DBI part can be written as

$$\det(g + F) = \det(\bar{g} + \bar{f}) \det(\mathbb{1} + X), \quad X = (\bar{g} + \bar{f})^{-1} (\hat{g} + f). \tag{A.4}$$

Hence the important step is to compute the components of matrix X in the expansion

$$\sqrt{\det(\mathbb{1} + X)} = 1 + \frac{1}{2} \text{Tr} X - \frac{1}{4} \text{Tr} X^2 + \frac{1}{8} (\text{Tr} X)^2 + \mathcal{O}(X^3). \tag{A.5}$$

The matrix $(\bar{g} + \bar{f})^{-1}$ can be written in a block diagonal form

$$(\bar{g} + \bar{f})^{-1} = \left(\frac{\mathcal{G}^{-1} + \mathcal{J}|_{0r}}{0} \middle| \begin{array}{c} 0 \\ \mathcal{G}^{ij} \end{array} \right), \tag{A.6}$$

where \mathcal{G}^{-1} and \mathcal{J} are its symmetric and antisymmetric parts, respectively. The non-zero elements of those matrices are

$$\mathcal{G}^{00} = -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 L^2 \sin^6 \alpha_n}, \quad \mathcal{G}^{rr} = \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{L^2 \sin^6 \alpha_n} r^2, \quad \mathcal{G}^{ij} = L^{-2} \tilde{g}^{ij} \tag{A.7}$$

and

$$\mathcal{J}^{0r} = -\mathcal{J}^{r0} = \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{L^2 \sin^6 \alpha_n}. \tag{A.8}$$

The matrix elements that contribute to the $\text{Tr} X$ are

$$\begin{aligned}X_0^0 &= -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 \sin^6 \alpha_n} \left\{ r^2 (\partial_0 \chi)^2 + \frac{q}{b^2} (\partial_0 \xi)^2 \right\}, \\ &\quad + \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \left\{ r^2 \partial_0 \chi \partial_r \chi + \frac{q}{b^2} \partial_0 \xi \partial_r \xi - L^{-2} f_{0r} \right\}, \\ X_r^r &= \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} r^2 \left\{ r^2 (\partial_r \chi)^2 + \frac{q}{b^2} (\partial_r \xi)^2 \right\} \\ &\quad - \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \left\{ r^2 \partial_0 \chi \partial_r \chi + \frac{q}{b^2} \partial_0 \xi \partial_r \xi + L^{-2} f_{0r} \right\}, \\ X_j^i &= r^2 \tilde{g}^{ik} \partial_k \chi \partial_j \chi + \frac{q}{b^2} \tilde{g}^{ik} \partial_k \xi \partial_j \xi + \xi (M^{(1)})_j^i + \xi^2 (M^{(2)})_j^i + L^{-2} \tilde{g}^{ik} f_{kj},\end{aligned}\tag{A.9}$$

where we have defined the matrices

$$\left(M^{(i)} \right)_j^i = \tilde{g}^{ik} \hat{g}_{kj}^{(i)}, \quad i = 1, 2. \quad (\text{A.10})$$

The metric \tilde{g}_{ij} corresponds to the five-dimensional space whose line element is given by (3.29). In order to calculate the trace of X^2 , we need to compute the non-diagonal elements of X up to first order in the fluctuations. The matrix elements that contribute to $\text{Tr} X^2$ are

$$\begin{aligned} X_r^0 &= -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 L^2 \sin^6 \alpha_n} f_{0r} \\ X_0^r &= -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{L^2 \sin^6 \alpha_n} r^2 f_{0r} \\ X_i^0 &= -\frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{r^2 L^2 \sin^6 \alpha_n} f_{0i} + \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{L^2 \sin^6 \alpha_n} f_{ri} \\ X_0^i &= L^{-2} \tilde{g}^{ij} f_{j0} \\ X_i^r &= \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{L^2 \sin^6 \alpha_n} r^2 f_{ri} - \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{L^2 \sin^6 \alpha_n} f_{0i} \\ X_r^i &= L^{-2} \tilde{g}^{ij} f_{jr} \end{aligned} \quad (\text{A.11})$$

Putting everything together we find that the $\text{Tr} X$ is given by

$$\begin{aligned} \text{Tr} X &= -2 \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} L^{-2} f_{0r} + 6 \cot \alpha_n \xi + \frac{1 + (3q - 1) \cos 2\alpha_n}{q \sin^2 \alpha_n} \xi^2 \\ &+ L^2 r^2 \mathcal{G}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{q}{b^2} L^2 \mathcal{G}^{\mu\nu} \partial_\mu \xi \partial_\nu \xi, \end{aligned} \quad (\text{A.12})$$

where we have used the following equations

$$\text{Tr} M^{(1)} = 6 \cot \alpha_n, \quad \text{Tr} M^{(2)} = \frac{1 + (3q - 1) \cos 2\alpha_n}{q \sin^2 \alpha_n}. \quad (\text{A.13})$$

The $\text{Tr}(X^2)$ is given by

$$\begin{aligned} \text{Tr}(X^2) &= 2L^{-4} \frac{\sin^6 \alpha_n + 2C_n(\alpha_n)^2}{\sin^6 \alpha_n} \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} f_{0r}^2 + 2\xi^2 \frac{1 + 3q + (3q - 1) \cos 2\alpha_n}{q \sin^2 \alpha_n} \\ &+ L^{-4} \left\{ 2 \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} \frac{f_{0i}^2}{r^2} - 2 \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^6 \alpha_n} r^2 f_{ri}^2 - f_{kj}^2 \right\}, \end{aligned} \quad (\text{A.14})$$

while for the $(\text{Tr}X)^2$ we have

$$\begin{aligned}
(\text{Tr}X)^2 &= 4L^{-4} \frac{\sin^6 \alpha_n + C_n(\alpha_n)^2}{\sin^{12} \alpha_n} C_n(\alpha_n)^2 f_{0r}^2 + 36 \cot^2 \alpha_n \xi^2 \\
&\quad - 24L^{-2} \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \cot \alpha_n \xi f_{0r}.
\end{aligned} \tag{A.15}$$

Putting everything together we calculate the expression of $\sqrt{\det(\mathbb{1} + X)}$

$$\begin{aligned}
\sqrt{\det(\mathbb{1} + X)} &= 1 - \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} L^{-2} f_{0r} + 3 \cot \alpha_n \xi + \frac{1}{2} L^2 r^2 \mathcal{G}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \\
&\quad + \frac{1}{2} \frac{q}{b^2} L^2 \mathcal{G}^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + \frac{3}{2} (2 \cot^2 \alpha_n - 1) \xi^2 + \frac{1}{4} \mathcal{G}^{\mu\rho} \mathcal{G}^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma} \\
&\quad - 3L^{-2} \frac{C_n(\alpha_n) \sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} \cot \alpha_n \xi f_{0r} + \dots
\end{aligned} \tag{A.16}$$

Since

$$\det(\bar{g} + \bar{f}) = - \frac{L^{14} \sin^6 \alpha_n}{\sin^6 \alpha_n + C_n(\alpha_n)^2} \cdot \det \tilde{g}, \tag{A.17}$$

the DBI part of the Lagrangian density is

$$\mathcal{L}_{DBI} = -T_6 \frac{\pi^2 N b^6 L^2}{q^2} \frac{\sin^3 \alpha_n}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} \sqrt{\tilde{g}} \sqrt{\det(\mathbb{1} + X)}, \tag{A.18}$$

where \tilde{g} is the determinant of (3.29)

$$\sqrt{\tilde{g}} = \frac{q^{3/2}}{8 b^5} \sin^3 \alpha_n \sin \theta_1 \sin \theta_2. \tag{A.19}$$

What remains is the computation of the WZ part. Using the conventions of section 3 we have

$$\mathcal{L}_{WZ} = -T_6 \frac{\pi^2 N}{8} \sin \theta_1 \sin \theta_2 C(\alpha) F_{0r}, \tag{A.20}$$

where $F_{0r} = \bar{f}_{0r} + f_{0r}$ and the function $C(\alpha)$ has to be expanded around α_n

$$C(\alpha) = C(\alpha_n) - 3 \sin^3 \alpha_n \xi - \frac{9}{2} \sin^2 \alpha_n \cos \alpha_n \xi^2 + \dots \tag{A.21}$$

Putting everything together, the WZ part becomes

$$\begin{aligned}
\mathcal{L}_{WZ} &= -T_6 \frac{\pi^2 N}{8} \sin \theta_1 \sin \theta_2 \left\{ \frac{L^2 C_n(\alpha_n)}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} C(\alpha_n) + C(\alpha_n) f_{0r} \right. \\
&\quad \left. - \frac{3L^2 \sin^3 \alpha_n C_n(\alpha_n)}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} \xi - 3 \sin^3 \alpha_n f_{0r} \xi - \frac{9}{2} L^2 \frac{\sin^2 \alpha_n \cos \alpha_n C_n(\alpha_n)}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}} \xi^2 \right\} + \dots
\end{aligned} \tag{A.22}$$

Finally, summing the DBI and WZ parts we obtain the result

$$\begin{aligned}
\mathcal{L}_{DBI} + \mathcal{L}_{WZ} = & -T_6 \frac{\pi^2 N b^6 L^2}{q^2} \frac{\sin^3 \alpha_n}{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)}} \sqrt{g} \left\{ 1 + \frac{\sqrt{q}}{b} \frac{C_n(\alpha_n)}{\sin^6 \alpha_n} C(\alpha_n) \right. \\
& + \left(\frac{\sqrt{q}}{b} C(\alpha_n) - C_n(\alpha_n) \right) \frac{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)^2}}{\sin^6 \alpha_n} L^{-2} f_{0r} \\
& + 3 \left(\cot \alpha_n - \frac{\sqrt{q}}{b} \frac{C_n(\alpha_n)}{\sin^3 \alpha_n} \right) \xi + \frac{1}{2} L^2 r^2 \mathcal{G}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{1}{2} \frac{q}{b^2} L^2 \mathcal{G}^{\mu\nu} \partial_\mu \xi \partial_\nu \xi \\
& + \left(\frac{3}{2} (2 \cot^2 \alpha_n - 1) - \frac{9}{2} \frac{\sqrt{q}}{b} \frac{\cos \alpha_n}{\sin^4 \alpha_n} C_n(\alpha_n) \right) \xi^2 + \frac{1}{4} \mathcal{G}^{\mu\rho} \mathcal{G}^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma} \\
& \left. - 3 L^{-2} \frac{\sqrt{\sin^6 \alpha_n + C_n(\alpha_n)}}{\sin^6 \alpha_n} \left(\frac{\sqrt{q}}{b} \sin^3 \alpha_n + C_n(\alpha_n) \cot \alpha_n \right) \xi f_{0r} \right\}. \tag{A.23}
\end{aligned}$$

In the above action the first two lines can be dropped containing either constant or linear or total derivative terms. The term linear in ξ also vanishes upon using the condition (3.21). The remaining terms constitute the action (4.2) used in the main text.

B Solution of the differential equation for $R(r)$

The differential equation (4.7) is of the form

$$\frac{d}{dr} \left(r^4 \frac{dR}{dr} \right) + (E^2 - A r^2) R = 0, \tag{B.1}$$

where A is a constant. To solve this differential equation we first study the asymptotic behavior of $R(r)$ at large r . Setting $R(r) \sim r^\lambda$ we end up with the condition

$$\lambda(\lambda + 3) = A \quad \implies \quad \lambda_\pm = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + A}. \tag{B.2}$$

Now that we know the asymptotic behavior of $R(r)$, we make the ansatz $R(r) = r^\lambda f(\frac{1}{r})$ and obtain the following differential equation for the function $f = f(u) = f(1/r)$

$$u^2 \frac{d^2 f}{du^2} - 2(\lambda + 1) u \frac{df}{du} + E^2 u^2 f = 0. \tag{B.3}$$

Now we observe that if we substitute $f(u) = u^{\lambda + \frac{3}{2}} g(u)$ the above equation becomes

$$u^2 \frac{d^2 g}{du^2} + u \frac{dg}{du} + (E^2 u^2 - t^2) g = 0, \tag{B.4}$$

where $t^2 = (\lambda + \frac{3}{2})^2 = \frac{9}{4} + A$, which is the Bessel differential equation with solution

$$g(u) = C_1 J_{\sqrt{\frac{9}{4}+A}}(Eu) + C_2 Y_{\sqrt{\frac{9}{4}+A}}(Eu). \quad (\text{B.5})$$

Finally, the solution for $R(r)$ is

$$R(r) = r^{-3/2} \left(C_1 J_{\sqrt{\frac{9}{4}+A}} \left(\frac{E}{r} \right) + C_2 Y_{\sqrt{\frac{9}{4}+A}} \left(\frac{E}{r} \right) \right), \quad (\text{B.6})$$

which has the correct asymptotic behavior at large r

$$R(r) = C_1 r^{-\frac{3}{2} + \sqrt{\frac{9}{4}+A}} + C_2 r^{-\frac{3}{2} - \sqrt{\frac{9}{4}+A}}. \quad (\text{B.7})$$

In our case the constant A appearing in (B.1) is

$$A = \frac{b^2}{q} \frac{l(l+2)}{\sin^2 \alpha_n} P. \quad (\text{B.8})$$

C The Laplacian on the five dimensional manifold M_5

The action of the Laplacian of the five-dimensional manifold M_5 , with metric \tilde{g}_{ij} (3.29) on a scalar depending on all the coordinates of M_5 is

$$\begin{aligned} \nabla_{M_5}^2 f &= \frac{c_1}{4 \sin^2 \frac{\alpha_n}{2}} \nabla_{S^3}^2 f + \frac{c_1 c_2}{2} \frac{1}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\theta_1} f) + \frac{c_1 c_2}{2} \frac{1}{\sin^2 \theta_1} \partial_{\phi_1}^2 f \\ &\quad - \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\theta_2} f) - \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_2} \partial_{\theta_2} (\sin \theta_2 \partial_{\theta_1} f) \\ &\quad - \frac{c_1 \cos(\phi_1 - \psi)}{\sin \theta_1 \sin \theta_2} \partial_{\theta_1} (\sin \theta_1 \partial_{\phi_2} f) - \frac{c_1 \cos(\phi_1 - \psi)}{\sin \theta_2} \partial_{\phi_2} \partial_{\theta_1} f \\ &\quad + \frac{c_1 \cos(\phi_1 - \psi) \cot \theta_2}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\psi} f) + c_1 \cot \theta_2 \partial_{\psi} (\cos(\phi_1 - \psi) \partial_{\theta_1} f) \\ &\quad - c_1 \cot \theta_1 \partial_{\phi_1} (\cos(\phi_1 - \psi) \partial_{\theta_2} f) - \frac{c_1 \cos(\phi_1 - \psi) \cot \theta_1}{\sin \theta_2} \partial_{\theta_2} (\sin \theta_2 \partial_{\phi_1} f) \\ &\quad + c_1 \frac{\cot \theta_1}{\sin \theta_2} \partial_{\phi_1} (\sin(\phi_1 - \psi) \partial_{\phi_2} f) + c_1 \frac{\cot \theta_1 \sin(\phi_1 - \psi)}{\sin \theta_2} \partial_{\phi_2} \partial_{\phi_1} f \\ &\quad + \frac{c_1}{\sin \theta_1 \sin \theta_2} \partial_{\phi_1} ((\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \partial_{\psi} f) \\ &\quad + \frac{c_1}{\sin \theta_1 \sin \theta_2} \partial_{\psi} ((\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \partial_{\phi_1} f) \end{aligned} \quad (\text{C.1})$$

where the last term is the Laplacian of the 3-sphere

$$\nabla_{S^3}^2 f = 4 \left\{ \frac{1}{\sin \theta_2} \partial_{\theta_2} (\sin \theta_2 \partial_{\theta_2} f) + \left(\frac{1}{\sin \theta_2} \partial_{\phi_2} - \cot \theta_2 \partial_{\psi} \right)^2 f + \partial_{\psi}^2 f \right\}, \quad (\text{C.2})$$

and the constants c_1, c_2 are given by

$$c_1 = \frac{b^2}{q} \frac{1}{\cos^2 \frac{\alpha_n}{2}}, \quad c_2 = 1 + q + (q - 1) \cos \alpha_n. \quad (\text{C.3})$$

Solving the corresponding eigenvalues problem in full generality seems very difficult. Instead, we look for configurations of the function f , in which the Laplacian acts on, that do not depend on all variables. We found two consistent truncations that simplify our eigenvalue problem and we present them in the following subsections.

C.1 Solutions of the form $f = \Omega(\theta_2, \phi_2, \psi)$

In the analysis that we presented in the previous sections of this paper we focused on configurations of the form $f = \Omega(\theta_2, \phi_2, \psi)$. For such configurations the above expression for the Laplacian becomes

$$\begin{aligned} \nabla_{M_5}^2 \Omega &= \frac{c_1}{4 \sin^2 \frac{\alpha_n}{2}} \nabla_{S^3}^2 \Omega - \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\theta_2} \Omega) \\ &\quad - \frac{c_1 \cos(\phi_1 - \psi)}{\sin \theta_1 \sin \theta_2} \partial_{\theta_1} (\sin \theta_1 \partial_{\phi_2} \Omega) + \frac{c_1 \cos(\phi_1 - \psi) \cot \theta_2}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\psi} \Omega) \\ &\quad - c_1 \cot \theta_1 \partial_{\phi_1} (\cos(\phi_1 - \psi) \partial_{\theta_2} \Omega) + c_1 \frac{\cot \theta_1}{\sin \theta_2} \partial_{\phi_1} (\sin(\phi_1 - \psi) \partial_{\phi_2} \Omega) \\ &\quad + \frac{c_1}{\sin \theta_1 \sin \theta_2} \partial_{\phi_1} ((\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \partial_{\psi} \Omega). \end{aligned} \quad (\text{C.4})$$

Notice that there are still derivatives to be taken with respect to the angles θ_1 and ϕ_1 as well as explicit dependence on these angles. Consistency of our truncation requires that all such dependencies drop out completely which indeed turns out to be the case as we are left with

$$\nabla_{M_5}^2 \Omega = \frac{c_1}{4 \sin^2 \frac{\alpha_n}{2}} \nabla_{S^3}^2 \Omega. \quad (\text{C.5})$$

In this way our eigenvalue problem is reduced to that of $\nabla_{S^3}^2$, a well known operator, which has eigenvalues $\lambda = -l(l + 2)$ where $l = 0, 1, 2, \dots$

C.2 Solutions of the form $f = \Omega(\theta_1, \phi_1)$

Another truncation that simplifies the eigenvalue problem is an ansatz of the form $f = \Omega(\theta_1, \phi_1)$. Then (C.1) simplifies as

$$\begin{aligned} \nabla_{M_5}^2 \Omega &= \frac{c_1 c_2}{2} \frac{1}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\theta_1} \Omega) + \frac{c_1 c_2}{2} \frac{1}{\sin^2 \theta_1} \partial_{\phi_1}^2 \Omega - \frac{c_1 \sin(\phi_1 - \psi)}{\sin \theta_2} \partial_{\theta_2} (\sin \theta_2 \partial_{\theta_1} \Omega) \\ &\quad - \frac{c_1 \cos(\phi_1 - \psi) \cot \theta_1}{\sin \theta_2} \partial_{\theta_2} (\sin \theta_2 \partial_{\phi_1} \Omega) + c_1 \cot \theta_2 \partial_{\psi} (\cos(\phi_1 - \psi) \partial_{\theta_1} \Omega) \\ &\quad + \frac{c_1}{\sin \theta_1 \sin \theta_2} \partial_{\psi} ((\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2 \sin(\phi_1 - \psi)) \partial_{\phi_1} \Omega) . \end{aligned} \quad (\text{C.6})$$

Again all dependence on θ_1, ϕ_1 and ψ , finally obtaining that

$$\nabla_{M_5}^2 \Omega = \frac{c_1 c_2}{2} \frac{1}{\sin \theta_1} \partial_{\theta_1} (\sin \theta_1 \partial_{\theta_1} \Omega) + \frac{c_1 c_2}{2} \frac{1}{\sin^2 \theta_1} \partial_{\phi_1}^2 \Omega = \frac{c_1 c_2}{2} \nabla_{S^2}^2 \Omega . \quad (\text{C.7})$$

We see that configurations of the form $f = f(\theta_1, \phi_1)$ lead to the well known eigenvalue problem of the Laplace operator on the unit S^2 , with eigenvalues $\lambda = -l(l+1)$, where $l = 0, 1, 2, \dots$

The results we have obtained for the fluctuations using the ansatz $f = \Omega(\theta_2, \phi_2, \psi)$ can be trivially extended to the case when $f = \Omega(\theta_1, \phi_1)$. We simply have to compare (C.5) and (C.7) and make the replacement

$$l(l+2) \rightarrow 4 \sin^2 \frac{\alpha_n}{2} \left(\sin^2 \frac{\alpha_n}{2} + q \cos^2 \frac{\alpha_n}{2} \right) l(l+1) . \quad (\text{C.8})$$

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